J. Fluid Mech. (1973) *vol.* 57, *part* 4, *pp.* 673–693

Printed in Great Britain

A generalized slender-body theory for fish-like forms

By J. N. NEWMAN
Massachusetts Institute of Technology, Cambridge, Massachusetts
AND T. Y. WU
California Institute of Technology, Pasadena, California

(Received 18 September 1972)

A consistent slender-body approximation is developed for the flow past a fish-like body with arbitrary combinations of body thickness and low-aspect-ratio fin appendages, but with the fins confined to the plane of symmetry of the body. Attention is focused on the interaction of the fin lifting surfaces with the body thickness, and especially on the dynamics of the vortex sheets shed from the fin trailing edges. This vorticity is convected by the (non-lifting) flow past the stretched-straight body, and departs significantly from the purely longitudinal orientation of conventional lifting-surface theory. Explicit results are given for axisymmetric bodies having fins with abrupt trailing edges, and calculations of the total lift force are presented for bodies with symmetric and asymmetric fin configurations, moving with a constant angle of attack.

1. Introduction

The analysis of flows past slender yawed bodies with lifting-surface appendages has been of interest to hydro- and aerodynamicists for a wide variety of applications. The treatments of this subject in aeronautical engineering are well documented (cf. Thwaites 1960), but that development of the theory is essentially limited to rigid-body motions and, more importantly, to configurations of the body and appended planar lifting surfaces wherein the vorticity shed from the trailing edges does not interact downstream with the body or with subsequent appended lifting surfaces; hence the lift distribution depends only on the local added-mass coefficient of the body cross-section, and appended planar lifting surfaces of low aspect ratio are included simply by computing their influence on the added-mass coefficient of the relevant body cross-section as, for example, in the case of a finned circle.

Interest in the theory of fish propulsion has led to a renewed activity in this branch of aerodynamics. Lighthill (1960) developed the unsteady slender-body theory for arbitrary lateral undulations of the body, in the absence of appended lifting surfaces and shed vorticity, and showed that the classical aerodynamic results remain valid in this case with a differential lift force proportional simply to the rate of change of the product of the added-mass coefficient and local body velocity, as measured in a fixed reference frame. This theory has been extended by Lighthill (1970), Wu (1971) and Wu & Newman (1972) to allow for a body

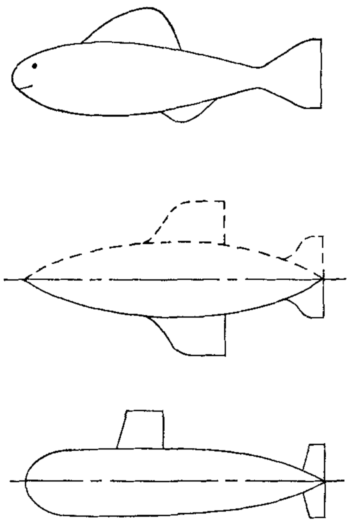

FIGURE 1. Geometrical configurations of a fish, yacht hull (with image above the free surface) and submarine.

with appended low-aspect-ratio lifting surfaces, situated in the centre-plane of symmetry normal to the lateral motions of the body, and with varying degrees of generality in the lifting-surface–body-thickness configuration. However, these generalizations are incomplete in the sense that either the body thickness is ignored, or alternatively it is treated in an inconsistent manner which does not account properly for the interaction of the 'sidewash' effect of the changing body form on the outboard trailing vortices shed from upstream appendages. It is our objective here to remove this deficiency, and to present a unified slender-body theory capable of embracing a wide variety of configurations, and subject only to the assumptions which follow consistently from slenderness, the neglect of viscous effects, and linearization of the lateral body motions. (An approach to the case of nonlinear motions has been outlined by Lighthill 1971.) We note also that the assumption of 'slenderness' requires not only that the body geometry is slowly varying in the longitudinal direction, but that its lateral motions do not change rapidly along the same direction; in other words, the wavelength of undulatory body motion must be large compared with the transverse dimensions of the body. This last restriction is discussed more explicitly by Lighthill (1970).

In addition to its application to the theory of fish propulsion, the present subject arises also in connexion with the horizontal manoeuvring of sailing yachts and submarines. For the yacht hull the keel corresponds to the side fin of the fish, and the rudder downstream to the fish caudal fin, as shown in figure 1. For this analogy to be valid, one must treat the free surface by a simple reflexion or image approach, so that the yacht's underbody and image constitute a symmetrical form moving in an otherwise unbounded fluid. A more detailed study along these lines, applying a preliminary version of the present theory, has been presented by Milgram (1972). The application to submarine manoeuvring is more obvious, since there is no consideration of the free surface required, but in this case it is generally necessary to consider the effects of asymmetry of the vertical lifting surfaces.

After a statement of the boundary-value problem in § 2, the general solution will be constructed in § 3. The Kutta condition of bounded velocity at the trailing edges plays a critical role in this solution, and is treated in § 4 for the special case of abrupt trailing edges, whereas the difficulties associated with a 'slant' trailing edge are outlined in § 5. The differential lift force acting on a transverse body section is derived in § 6, and in § 7 computations of the total lift force are presented for a body having axisymmetric thickness and abrupt trailing edges.

2. The boundary-value problem

Cartesian co-ordinates (x, y, z) are employed, with the x axis coincident with the longitudinal body axis in the 'stretched-straight' position of steady forward motion and the origin fixed with respect to the mean position of the body. Hence, with the body nose at $x = -l_N$ and tail at $x = l_T$, a steady streaming flow with components $(U, 0, 0)$ is incident upon the body, as shown in figure 2(*a*) below. We assume the body to be symmetrical about the (vertical) plane of symmetry x, y, with thickness $2g(x, y)$, but, in general, asymmetric about the horizontal plane x, z. Planar appendages are situated in the vertical plane of symmetry, above and/or beneath the body 'hull' or 'fuselage' where the thickness is distributed. The projection of the body with appendages on the x, y plane will be denoted by the curves $y = -b_1(x)$ and $y = b_2(x)$.

The unsteady motions of the body are described by a lateral displacement $z = h(x, t)$ and, with the usual assumptions of ideal flow, the fluid motions by the (positive) gradient of a velocity potential $\phi(x, y, z, t)$ which is governed by Laplace's equation in the fluid domain. This potential satisfies a kinematic boundary condition on the body, a dynamic boundary condition on the vortex sheets, and tends to the free-stream potential Ux at large distances away from the body and its trailing vortices. In addition, we must impose a Kutta condition proscribing unbounded fluid velocity components at the trailing edges.

Before developing the above boundary conditions in detail we note, following Lighthill (1960), that "in this problem it is almost essential to make a transformation of co-ordinates, so that the body becomes a fixed surface – for, otherwise, there are severe difficulties due to applying boundary conditions at a surface whose position is displaced in a direction in which gradients are specially steep". In essence, the difficulty stems from the fact that the unsteady normal velocity on the body is affected not only by the velocity of the body, but also by its displacement in the steady-state velocity field. This problem has been discussed in detail by Timman & Newman (1962), who derive a consistent linearized kinematic boundary condition to be applied on the *mean* position of the body surface in the Eulerian reference frame (x, y, z). However, that approach is cumbersome, and for the problem at hand we follow Lighthill (1960) and use 'stretched-straight' co-ordinates (X, Y, Z) which are fixed in the body and defined by the transformation

$$X = x, \quad Y = y, \quad Z = z - h(x, t), \quad T = t, \tag{2.1}$$

so that

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial X}-h_x\frac{\partial}{\partial Z},\quad \frac{\partial}{\partial y}=\frac{\partial}{\partial Y},\quad \frac{\partial}{\partial z}=\frac{\partial}{\partial Z},\quad \frac{\partial}{\partial t}=\frac{\partial}{\partial T}-h_t\frac{\partial}{\partial Z}. \tag{2.2}$$

The velocity potential Φ is defined by the transformation

$$\Phi(X,Y,Z,T)=\phi(x,y,z,t) \tag{2.3}$$

and from (2.2) the Laplace equation becomes

$$\left(\frac{\partial}{\partial X}-h_X\frac{\partial}{\partial Z}\right)(\Phi_X-h_X\Phi_Z)+\Phi_{YY}+\Phi_{ZZ}=0. \tag{2.4}$$

The body surface S_b may be defined by the equation

$$z-h(x,t)\mp g(x,y)=0\quad\text{on}\quad S_b, \tag{2.5}$$

where the $\mp$ signs correspond to the two sides of the body. Thus, in transformed co-ordinates,

$$F(X,Y,Z)\equiv Z\mp g(X,Y)=0\quad\text{on}\quad S_b, \tag{2.6}$$

and the kinematic boundary condition may be written as $DF/Dt=0$, where D/Dt is the substantial derivative $(\partial/\partial t+\nabla\phi.\nabla)$. Transforming this operator gives the kinematic boundary condition

$$-h_TF_Z+(\Phi_X-h_X\Phi_Z)(F_X-h_XF_Z)+\Phi_YF_Y+\Phi_ZF_Z=0\quad\text{on}\quad S_b. \tag{2.7}$$

Finally, we write Bernoulli's equation for the pressure in the form

$$\begin{aligned}p-p_\infty&=-\rho\phi_t-\tfrac{1}{2}\rho[\nabla\phi.\nabla\phi-U^2]\\&=-\rho(\Phi_T-h_T\Phi_Z)-\tfrac{1}{2}\rho[(\Phi_X-h_X\Phi_Z)^2+\Phi_Y^2+\Phi_Z^2-U^2].\end{aligned} \tag{2.8}$$

Before proceeding further we invoke the dual assumptions that (i) the body is slender and (ii) the lateral motion $h(x,t)$ is a small perturbation of the stretched-straight motion $h=0$. From the slenderness assumption, say $\delta\ll 1$ is a parameter such that, in the inner region near and on the body surface, $(Y,Z)=O(\delta)$, whereas $X=O(1)$. It follows that, in this same region, $(\partial/\partial Y,\partial/\partial Z)=O(\delta^{-1})$, whereas $\partial/\partial X=O(1)$. From the second assumption, we write the potential in the form

$$\Phi=UX+\Phi_0(X,Y,Z)+\Phi_1(X,Y,Z,T), \tag{2.9}$$

where $UX+\Phi_0$ is the potential for steady flow past the stretched-straight body and Φ_1 is the lateral-motion contribution. From classical slender-body theory we note that

$$\Phi_0=O(\delta^2\log\delta), \tag{2.10}$$

whereas we anticipate that

$$\Phi_1=O(\delta h). \tag{2.11}$$

(This estimate can be verified *a posteriori*.)

Substituting in Laplace's equation (2.4) it follows that, to leading order,

$$\Phi_{0YY}+\Phi_{0ZZ}=0, \tag{2.12}$$

$$\Phi_{1YY}+\Phi_{1ZZ}=0, \tag{2.13}$$

where the error is a factor $1+O(\delta^2 \log \delta, \delta h, h^2)$ in the first equation and a factor $1+O(\delta^2, \delta h, h^2)$ in the second. Proceeding in a similar manner, the kinematic boundary condition (2.7) gives, for the stretched-straight case $h = 0$,

$$\Phi_{0Y} F_Y + \Phi_{0Z} F_Z + U F_X = 0 \quad \text{on} \quad S_b \tag{2.14}$$

and, subtracting this from (2.7) and retaining linear terms in h,

$$\Phi_{1Y} F_Y + (\Phi_{1Z} - h_T - U h_X) F_Z = 0 \quad \text{on} \quad S_b \tag{2.15}$$

as the boundary condition for Φ_1.

As observed by Lighthill (1960), equation (2.15), with the Laplace equation (2.13), is the boundary-value problem for an unsteady potential Φ_1 due to motions of a two-dimensional cylinder, whose cross-section Σ_b is the intersection of S_b with the plane X = constant, and which moves in the Z direction with velocity

$$V(X, T) = h_T + U h_X \tag{2.16}$$

subject to the condition that $\Phi_1 \to 0$ at large distances from the cylinder. However, in the present problem an additional dynamic boundary condition is required on the vortex sheets, and for this purpose we note from Bernoulli's equation (2.8) that

$$\begin{aligned} p - p_\infty &= -\rho[U\Phi_{0X} + \tfrac{1}{2}(\Phi_{0Y}^2 + \Phi_{0Z}^2)] - \rho[\Phi_{1T} + U\Phi_{1X} + \Phi_{0Y}\Phi_{1Y} \\ &\quad + \Phi_{0Z}(\Phi_{1Z} - V)] + O(h^2, \delta^4) \\ &\equiv p_0 + p_1, \end{aligned} \tag{2.17}$$

say. Here p_0 is the linearized stretched-straight pressure, and p_1 the leading-order lateral-motion pressure. For the latter it should be emphasized that the cross-flow products $\Phi_{0Y}\Phi_{1Y} + \Phi_{0Z}(\Phi_{1Z} - V)$ are of comparable order to $D\Phi_1 \equiv \Phi_{1T} + U\Phi_{1X}$, so that in this way there is an interaction between the two flow fields which will affect, in particular, the dynamics of the trailing vorticity.

In view of the symmetry of S_b about the Z axis, it follows from (2.14) that Φ_0 will be an even function of the variable Z, whereas from (2.15) Φ_1 will be an odd function of Z. Hence the pressure p_1 is odd and from continuity of pressure across the vortex sheet wake S_w, $p_1 = 0$ on S_w. Noting that Φ_{0Z} vanishes on this plane, we obtain the condition of continuous pressure across S_w in the form

$$D\Phi_1 + \Phi_{0Y}\Phi_{1Y} = 0 \quad \text{on} \quad S_w \tag{2.18}$$

and, since Φ_1 must be continuous outside S_w,

$$\Phi_1 = 0 \quad \text{on} \quad S_c, \tag{2.19}$$

where S_c is the complementary portion of the plane $Z = 0$ outside $S_b + S_w$. Equation (2.18) is the appropriate dynamic boundary condition on S_w, and it can be interpreted physically by the statement that vorticity is convected by the steady flow components $(U, \Phi_{0Y}, 0)$. Thus Φ_1 is constant in a reference frame moving with these velocity components or, mathematically, (2.18) can be integrated along the characteristics (U, Φ_{0Y}) in the X, Z plane to yield the condition

$$\Phi_1(X, Y, 0\pm, T) = \Phi_1(X_*, Y_*, 0\pm, T_*) \quad \text{on} \quad S_w. \tag{2.20}$$

Here X_* and Y_* denote (upstream) co-ordinates of the steady streamline passing through (X, Y) – for example, at the trailing edge – and T_* is the retarded time:

$$T_* = T-(X-X_*)/U. \tag{2.21}$$

Thus, for the problem at hand, the trailing vortices are convected along the stretched-straight streamlines, and the retarded potential Φ_* must be defined in accordance with this fact. We emphasize that this interaction between the body thickness and the vortex sheets is a consequence of the consistent slender-body approximation of Bernoulli's equation, and it is *not* valid to assume that the linearized vortex filaments will be parallel to the X axis, unless the body thickness is zero or locally independent of X. The treatment of this 'sidewash' effect is the principal contribution of the present paper, by comparison with the earlier paper of Wu & Newman (1972), where for the sake of expediency the cross-flow terms in Bernoulli's equation were neglected.

The remaining conditions to be imposed are the Kutta condition

$$\nabla\Phi_1 < \infty \quad \text{at trailing edges} \tag{2.22}$$

and

$$(\Phi_0, \Phi_1) \to 0 \quad \text{for} \quad |Y+iZ| \to \infty \quad \text{or} \quad X \to -\infty. \tag{2.23}$$

The last condition states that the body disturbance vanishes at infinity, except possibly in the downstream direction $X \to \infty$ where, in general, trailing vorticity will be present. This completes the statement of the linearized boundary-value problem for Φ_1. Since, at this point, only first-order quantities are involved in the unsteady parameter h, we can consistently replace (X, Y, Z, T) by the former co-ordinates (x, y, z, t), and the corresponding potentials by

$$\phi_0(x,y,z) \equiv \Phi_0(X, Y, Z) \tag{2.24}$$

and

$$\phi_1(x,y,z,t) \equiv \Phi_1(X, Y, Z, T). \tag{2.25}$$

Moreover, since the potential satisfies the two-dimensional Laplace equation (2.13), in the transverse 'cross-flow' planes x = constant, it then follows that

$$\phi_1(x,y,z,t) = \mathrm{Re}\,(\phi+i\psi) \equiv \mathrm{Re}\,(f), \tag{2.26}$$

where the complex potential $f(\zeta, x, t)$ is an analytic function of the complex variable $\zeta = y+iz$. We note from symmetry that $\mathrm{Re}\,[f(\zeta)] = -\mathrm{Re}\,[f(\bar{\zeta})]$ and hence

$$f(\bar{\zeta}) = -\overline{f(\zeta)}, \tag{2.27}$$

where the bars denote complex conjugates. Finally, the boundary condition (2.15) can be expressed in the form

$$\partial(\phi - Vz)/\partial n = 0 \quad \text{on} \quad S_b \tag{2.28}$$

and hence, from the Cauchy–Riemann equations,

$$\psi(y,z,x,t) = -V(x,t)\,y+\psi_0(x,t) \quad \text{on} \quad S_b, \tag{2.29}$$

where the constant of integration ψ_0 is arbitrary. Thus by (2.26) $\phi_1 = \mathrm{Re}\,(f)$, where the real part of the complex potential f is specified on the plane of symmetry outside the body [equations (2.19) and (2.20)], and the imaginary part is specified

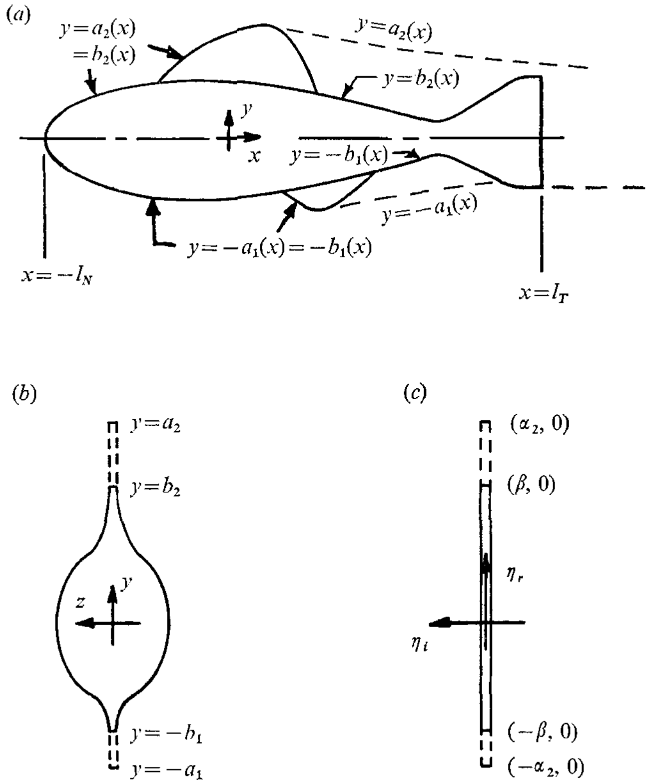

FIGURE 2. (a) The plan-form of a finned slender body, showing the tip vortices situated at $y = -a_1(x)$ and $y = a_2(x)$. (b) The cross-flow plane in physical co-ordinates. (c) Mapped co-ordinates; in both cases the vortex sheets are indicated by dashed lines.

on the body by equation (2.29). This Riemann–Hilbert problem is completed by the statement (2.22) of the Kutta condition and of vanishing of the potential at infinity (2.23).

3. Solution of the boundary-value problem

The body cross-section Σ_b at each station x will consist, in general, of a thickness portion and planar fins, situated on the real axis of the complex plane $\zeta = y + iz$. The projection of the contour Σ_b on the y axis is the segment $-b_1(x) \leqslant y \leqslant b_2(x)$. Downstream of the trailing edges, trailing vortex sheets will be situated outboard of Σ_b on the y axis, extending from the body at $\mp b_i(x)$ to the tip vortices situated at the points $\mp a_i(x)$. In these regions the parameters $a_i(x)$ are the y co-ordinates of the stretched-straight streamlines $(U, \phi_{0y}, 0)$ which coincide with the fin tips at maximum span points, and hence $y = \mp a_i(x)$ are the positions of the two tip vortices in the x, y plane. It is convenient for the subsequent analysis to extend this definition of $a_i(x)$ by stating that, in leading-edge regions where there is no upstream trailing vortex sheet, and also downstream if the leading-edge span is greater than the upstream tip vortex, $a_i(x) \equiv b_i(x)$. Hence the contours

$y = -a_1(x)$ and $y = a_2(x)$ define the boundaries of the body–vortex-sheet combination in the x, y plane, as shown in figure 2.

To proceed further it is expedient to map the entire fluid domain outside Σ_b onto a complex plane $\eta = \eta_r + i\eta_i$, with the body contour mapped onto the slit $-\beta(x) < \eta_r < \beta(x)$ and the remaining portions of the y axis onto $\eta_i = 0$, $|\eta_r| > \beta$. The body-plus-vortex-sheet boundaries $\mp a_i(x)$ are then mapped onto the curves $\eta_r = \mp \alpha_i(x)$, and on leading-edge regions without trailing vortices $\alpha_i = \beta$. The conformal transformation is defined as

$$\zeta = \zeta(\eta; x) \quad \text{or} \quad \eta = \eta(\zeta; x), \tag{3.1}$$

where $\zeta(\eta; x)$ is an analytic function of η for every x, satisfying the conditions that

$$d\zeta/d\eta \to 1 \quad \text{as} \quad |\zeta| \to \infty. \tag{3.2}$$

From the definitions of $\mp \alpha_i$ and β it follows that

$$\mp \beta(x) = \eta(\mp b_i(x); x) \quad (i = 1, 2) \tag{3.3}$$

and

$$\mp \alpha_i(x) = \eta(\mp a_i(x); x) \quad (i = 1, 2). \tag{3.4}$$

We note that for simplicity in the subsequent equations the body has been mapped onto a symmetrical slit, which generally implies a translation of the origin, so that the image of the x axis will *not* be a straight line in the x, η_r plane, unless the body is symmetrical. Finally (3.2) implies that

$$\eta = \zeta + \sum_{n=0}^{\infty} c_n(x)\, \zeta^{-n} \quad \text{as} \quad |\zeta| \to \infty, \tag{3.5}$$

where the coefficients $c_n(x)$ are real by virtue of the body symmetry in z.

It is convenient to keep the complex potential $f = \phi + i\psi$ invariant under the conformal transformation (3.1), i.e. $f(\zeta) = f(\zeta(\eta))$. The kinematic boundary condition on the body surface then follows from (2.27), and may be transformed directly into the mapped-plane variables. However, the dynamic boundary condition for ϕ following from (2.20) is

$$\phi(y \pm i0, x, t) = \phi(y_* \pm i0, x_*, t_*) \equiv \pm \phi_*(y, x, t) \quad \text{on} \quad S_w. \tag{3.6}$$

Here the retarded potential ϕ_* is the value of ϕ at the retarded time t_*, but at that point on the trailing edge with co-ordinates (x_*, y_*) which coincides with the same streamline $(U, \phi_{0y}, 0)$ as the field point $(x, y, 0)$. (Note that $y_* = -b_1(x_*)$ on the lower trailing edge, and $y_* = b_2(x_*)$ on the upper trailing edge.)

The boundary conditions on the complex potential can now be expressed in terms of a Riemann–Hilbert problem and stated as follows: on the real axis $\eta_i = 0\pm$,

$$\mathrm{Re}\,(f) = \begin{cases} 0 & \text{for} \quad -\infty < \eta_r < -\alpha_1(x), & (3.7) \\ \pm \phi_*(\eta_r, x, t) & \text{for} \quad -\alpha_1(x) < \eta_r < -\beta(x), & (3.8) \end{cases}$$

$$\mathrm{Im}\,(f) = -Vy(\eta_r) + \psi_0(x, t) \quad \text{for} \quad -\beta(x) < \eta_r < \beta(x), \tag{3.9}$$

$$\mathrm{Re}\,(f) = \begin{cases} \pm \phi_*(\eta_r, x, t) & \text{for} \quad \beta(x) < \eta_r < \alpha_2(x), & (3.10) \\ 0 & \text{for} \quad \alpha_2(x) \leqslant \eta_r < \infty. & (3.11) \end{cases}$$

The boundary conditions (3.7)–(3.11) must be supplemented by the Kutta condition on trailing edges and by the condition $f \to 0$ as $|\zeta| \to \infty$.

The solution of this problem in the absence of vortex sheets, i.e. when $\alpha_i = \beta$, is the 'elementary' function

$$f \equiv f_e = iV[(\eta^2-\beta^2)^{\frac{1}{2}}-\zeta-c_0(x)], \tag{3.12}$$

since (3.12) satisfies (3.7), (3.9) and (3.11) and by virtue of (3.5) vanishes at infinity. When vortex sheets are present, a complementary solution f_c must be added to give the total potential

$$f = f_e+f_c, \tag{3.13}$$

where f_c satisfies the following conditions on $\eta_i = 0$:

$$\operatorname{Re}(f_c) = \begin{cases} 0 & \text{for} \quad -\infty < \eta_r < -\alpha_1, & (3.14)\\ \phi_* & \text{for} \quad -\alpha_1 < \eta_r < -\beta, & (3.15)\end{cases}$$

$$\operatorname{Im}(f_c) = \psi_{c0}(x,t) \quad \text{for} \quad -\beta < \eta_r < \beta, \tag{3.16}$$

$$\operatorname{Re}(f_c) = \begin{cases} \phi_* & \text{for} \quad \beta < \eta_r < \alpha_2, & (3.17)\\ 0 & \text{for} \quad \alpha_2 < \eta_r < \infty. & (3.18)\end{cases}$$

Using Hilbert transform techniques (cf. Muskhelishvili 1953, p. 92), a particular solution of (3.14)–(3.18) is

$$f_c(\eta,x,t) = -\frac{i}{\pi}(\eta^2-\beta^2)^{\frac{1}{2}}\left(\int_{-\alpha_1}^{-\beta}+\int_{\beta}^{\alpha_2}\right)d\eta_1\frac{\phi_*(\eta_1,x,t)}{(\eta_1^2-\beta^2)^{\frac{1}{2}}(\eta_1-\eta)}+i\psi_{c0}(x,t), \tag{3.19}$$

and this solution will vanish at infinity if

$$\psi_{c0}(x,t) = -\frac{1}{\pi}\left(\int_{-\alpha_1}^{-\beta}+\int_{\beta}^{\alpha_2}\right)d\eta_1\frac{\phi_*(\eta_1,x,t)}{(\eta_1^2-\beta^2)^{\frac{1}{2}}}. \tag{3.20}$$

The solution defined by (3.12), (3.13), (3.19) and (3.20) is valid for all body regions, since the complementary function f_c as defined by (3.19) and (3.20) vanishes identically on leading-edge regions where $\alpha_i = \beta$. In regions of trailing vorticity, the complementary solution is non-zero, and is determined entirely by the retarded potential ϕ_* at the upstream trailing edges. It remains, however, to determine the value of this parameter so as to satisfy the Kutta condition on the trailing edges. To this end a distinction must be made between 'abrupt' trailing edges and 'slant' trailing edges, the former being perpendicular to the x axis, and hence representing a step-function reduction of the span, whereas the slant trailing edges are defined as all other cases involving a continuous reduction in span (relative to the width of the trailing vortex filament) with increasing downstream distance. The abrupt case is relatively straightforward to analyse,† and will be treated in § 4; the difficulties associated with the slant-trailing-edge geometry are described in § 5. We emphasize here that for the present purposes a trailing edge is a sharp-finned or cusped region of the body contour, the span of

† In this case a weak Kutta condition is imposed, by requiring that the velocity potential be continuous at the trailing edge. As in the analogous case of low-aspect-ratio planar foils, this will result in a local singularity of the pressure, unless the semi-spans $b_i(x)$ and body thickness are locally stationary at the trailing edge.

which is decreasing (strictly speaking, relative to the span of the stretched-straight streamlines), whereas there may also occur a region where the body contour is smooth, without sharp edges, and contracting in thickness as along the afterbody of a smooth body without fins. In the latter case of a smooth body contour, the imposition of a Kutta condition is not required, since the cross-flow velocity (in the physical plane) will be finite by virtue of the local zero in the derivative of the mapping function at these points.

4. The case of abrupt trailing edges

At an abrupt trailing edge, perpendicular to the x axis, the Kutta condition is satisfied by requiring that the potential be continuous across the trailing edge. The elementary or leading-edge solution f_e, defined by (3.12), will be discontinuous, however, as a result of the step-function discontinuity in the parameter $\beta(x)$, and this discontinuity must be absorbed in the complementary solution f_c. From the dynamic boundary conditions (3.15) and (3.17) it is clear that this will be true, provided that the retarded potential ϕ_* at the trailing edge increases by an amount equal and opposite to the decrease in the real part of f_e. Noting that $\mathrm{Re}(f_e) = 0$ downstream of the trailing edge, outboard of the body slit $(-\beta, \beta)$, it follows that ϕ_* may be determined at the *first* trailing edge by the equation

$$\phi_*(\eta_*, x_*, t_*) = \lim_{x \to x_* - 0} - V(x, t_*)(\beta^2 - \eta_*^2)^{\frac{1}{2}} \equiv - V(x_*, t_*)(\beta_*^2 - \eta_*^2)^{\frac{1}{2}}, \quad (4.1)$$

where β_* denotes the value of β immediately upstream of the trailing-edge discontinuity. For the symmetric case, where both opposite trailing edges are at the same value of $x \equiv x_*$, equation (4.1) remains valid.

At subsequent abrupt trailing edges one must use the more general relation

$$\begin{aligned}\phi_*(\eta_*, x_*, t_*) &= \lim_{x \to x_* - 0} \mathrm{Re}(f_e + f_c) \\ &= - V(x_*, t_*)(\beta_*^2 - \eta_*^2)^{\frac{1}{2}} + \mathrm{Re} f_c(\eta_*, x_*, t_*).\end{aligned} \quad (4.2)$$

Equation (3.6) may be rewritten in the mapped variables as

$$\phi_*(\eta, x, t) = \phi_*(\eta_*, x_*, t_*) \quad \text{for} \quad -\alpha_1 < \eta < -\beta, \quad \beta < \eta < \alpha_2, \quad (4.3)$$

with (η, x) and (η_*, x_*) defined to lie on the same streamline of the stretched-straight flow. Equations (4.1)–(4.3) then serve to determine the unknown ϕ_* in (3.19) and (3.20) and the solution for f_c is complete.

5. Slant trailing edges

On a slant trailing edge, $b_i(x)$ and $\beta(x)$ are continuous, generally decreasing, functions of x. In this case ϕ_* must be determined so that the complex velocity $f'(\zeta)$ is bounded at $\mp b_i$, or such that, as the trailing edge(s) are approached, say on the positive side of the foil $z = 0+$,

$$\mathrm{Re} f(\eta, x, t) = -\phi_*(\mp \beta, x, t) + O(\beta - |\eta|). \quad (5.1)$$

In particular, the solution (3.19) for f_c must be determined such that there is no term in (5.1) proportional to $(\eta \pm \beta)^{\frac{1}{2}}$. From the theory of Hilbert transforms (Muskhelishvili 1953, pp. 73–4) it is well known that the function f_c, as defined by (3.19), tends to the limiting value

$$\operatorname{Re}[f_c(\eta, x, t)] \to -\phi_*(\mp\beta, x, t) + o(1) \tag{5.2}$$

on the upper side of the cut $|\eta| < \beta$, as $\eta \to \pm\beta$. However, this estimate is not sufficient to satisfy the Kutta condition (5.1), and ϕ_* must now be suitably chosen to ensure that the $o(1)$ remainder in (5.2) cancels the square-root term in f_e, so that the total potential satisfies (5.1).

First, let us consider the limiting value of f_c as given by (3.19), at the lower trailing edge, as $\zeta \to -\beta + 0$. After adding and subtracting the quantity $\phi_*(-\beta)$ in the integrand, it follows that

$$\begin{aligned} f_c = &-\frac{i}{\pi}(\eta^2-\beta^2)^{\frac{1}{2}}\left(\int_{-\alpha_1}^{-\beta}+\int_{\beta}^{\alpha_2}\right)d\eta_1\frac{\phi_*(\eta_1)-\phi_*(-\beta)}{(\eta_1^2-\beta^2)^{\frac{1}{2}}(\eta_1-\eta)} \\ &-\frac{i}{\pi}(\eta^2-\beta^2)^{\frac{1}{2}}\phi_*(-\beta)\left(\int_{-\infty}^{-\beta}-\int_{-\infty}^{-\alpha_1}+\int_{\beta}^{\infty}-\int_{\alpha_2}^{\infty}\right)\frac{d\eta_1}{(\eta_1^2-\beta^2)^{\frac{1}{2}}(\eta_1-\eta)} \\ &+i\psi_{c0}(x,t), \end{aligned} \tag{5.3}$$

where the (regular) term ψ_{c0} is given by (3.20). On the second line of (5.3), the integrals over the domains $(-\infty, -\beta)$ and (β, ∞) may be evaluated (cf. Erdélyi 1954, equation 15.2 (23)) to yield the quantity

$$\pi(\beta^2-\eta^2)^{-\frac{1}{2}} \quad \text{for} \quad |\eta| < \beta.$$

Hence this contribution to (5.3) gives precisely the leading-order regular term ϕ_* in (5.1) and (5.2), and it is the remainder of (5.3) which must be examined. But in the remaining integrals the limit $\eta \to -\beta$ is regular and hence it follows that

$$\begin{aligned} f_c = &-\frac{i}{\pi}(\eta^2-\beta^2)^{\frac{1}{2}}\left\{\left(\int_{-\alpha_1}^{-\beta}+\int_{\beta}^{\alpha_2}\right)d\eta_1\frac{\phi_*(\eta_1)-\phi_*(-\beta)}{(\eta_1-\beta)^{\frac{1}{2}}(\eta_1+\beta)^{\frac{3}{2}}}\right. \\ &\left.-\phi_*(-\beta)\left(\int_{-\infty}^{-\alpha_1}+\int_{\alpha_2}^{\infty}\right)\frac{d\eta_1}{(\eta_1-\beta)^{\frac{1}{2}}(\eta_1+\beta)^{\frac{3}{2}}}\right\}+\text{regular terms}+O(\eta+\beta), \end{aligned} \tag{5.4}$$

where the regular terms include contributions from ϕ_* and ψ_{c0}. Adding the square-root term from the expression (3.12) for f_c, it follows that the Kutta condition will be satisfied at $\zeta = -\beta$ if, and only if,

$$0 = \pi V - \{\text{terms in braces in (5.4)}\}. \tag{5.5}$$

Hence the Kutta condition has been stated as an integral equation for ϕ_* along the trailing edge. By taking advantage of the expression

$$\frac{d}{d\eta_1}\left(\frac{\eta_1-\beta}{\eta_1+\beta}\right)^{\frac{1}{2}} = \beta(\eta_1-\beta)^{-\frac{1}{2}}(\eta_1+\beta)^{-\frac{3}{2}} \tag{5.6}$$

and integrating by parts, (5.4) and (5.5) can be replaced by the simpler integro-differential equation

$$0 = \pi V\beta + \left(\int_{-\alpha_1}^{-\beta}+\int_{\beta}^{\alpha_2}\right)\left(\frac{\eta_1-\beta}{\eta_1+\beta}\right)^{\frac{1}{2}}\frac{d\phi_*}{d\eta_1}d\eta_1, \tag{5.7}$$

where the condition $\phi_*(\mp\alpha_i) = 0$ has been used from (3.7) and (3.11). Performing a similar analysis for the opposite trailing edge at $\eta = +\beta$ gives the corresponding equation

$$0 = \pi V\beta - \left(\int_{-\alpha_1}^{-\beta} + \int_{\beta}^{\alpha_2}\right)\left(\frac{\eta_1+\beta}{\eta_1-\beta}\right)^{\frac{1}{2}} \frac{d\phi_*}{d\eta_1} d\eta_1. \tag{5.8}$$

Finally, adding and subtracting (5.7) and (5.8) gives the alternative pair of equations

$$\begin{pmatrix}\pi V\\ 0\end{pmatrix} = \left(\int_{-\alpha_1}^{-\beta} + \int_{\beta}^{\alpha_2}\right)\begin{pmatrix}1\\ \eta_1\end{pmatrix}\frac{d\phi_*}{d\eta_1}\frac{d\eta_1}{(\eta_1^2-\beta^2)^{\frac{1}{2}}}. \tag{5.9}$$

These two integro-differential equations must be satisfied, but subject to the additional constraint that $\phi_*(\eta, x, t)$ varies, as a function of x and t, so as to satisfy the dynamic condition (3.6). The only case where a closed-form solution of (5.9) appears feasible is that of an uncambered planar foil at constant angle of attack, with zero thickness and with symmetric trailing edges. Then, with $\alpha_1 = \alpha_2 =$ constant, (5.9) can be reduced to an Abel equation with the solution

$$\phi_*(\eta, x, t) = -V(\alpha^2-\eta^2)^{\frac{1}{2}}.$$

For the more general case of a planar foil, with asymmetric trailing edges and arbitrary $V(x,t)$, an integral expression for ϕ_* can be inferred from Wu & Newman (1972, equation 7.30).

6. The differential lift force

The lift force can be obtained directly from pressure integration over the body surface, but a simpler approach is to use momentum conservation so as to relate the differential lift force to the asymptotic form of the velocity potential at large distances away from the body cross-section Σ_b. Here we follow Lighthill (1960, appendix) in adopting the latter approach, and show that his analysis is readily generalized to treat the case where lifting surfaces and trailing vortex sheets are present.

As in § 2, we use 'stretched-straight' co-ordinates $X = x$, $Y = y$, $Z = z-h(x,t)$ and $T = t$ in order to perform the pressure integration over the oscillating surface of the body. The linearized differential lift force follows from (2.17) in the form

$$\mathscr{L}(X) = \int_{\Sigma_b} p_1\, dY = -\rho\oint_{\Sigma_b+\Sigma_w}[\Phi_{1T} + U\Phi_{1X} + \Phi_{0Y}\Phi_{1Y} + \Phi_{0Z}(\Phi_{1Z}-V)]\, dY, \tag{6.1}$$

where the contour integral has been extended to include the vortex wake contours Σ_w, since the pressure is zero on Σ_w and no contribution to the integral in (6.1) results from that change.

Now consider the 'generalized added-mass' integral

$$A(X,T) = \oint_{\Sigma_b+\Sigma_w}\Phi_1\, dY = \oint_{\Sigma_b+\Sigma_{w0}}\Phi_1\, dY, \tag{6.2}$$

where Σ_{w0} is the extension of Σ_w out to the maximum span co-ordinate, the width of Σ_{w0} being independent of X (Φ_1 vanishes on Σ_{w0} outside Σ_w). Then

$$\begin{aligned} DA &= A_T + UA_X \\ &= \oint_{\Sigma_b+\Sigma_{w0}} [\Phi_{1T} + U\Phi_{1X} + U\Phi_{1Z}(\partial Z/\partial X)_{Y=\text{const}}]\,dY. \end{aligned} \tag{6.3}$$

Noting that, on Σ_b, $F_X\,dX + F_Y\,dY + F_Z\,dZ = 0$, it follows that

$$\begin{aligned} U(\partial Z/\partial X)_{Y=\text{const}}\,dY &= -U(F_X/F_Z)\,dY = [\Phi_{0Z} + \Phi_{0Y}F_Y/F_Z]\,dY \\ &= \Phi_{0Z}\,dY - \Phi_{0Y}\,dZ \quad \text{on} \quad \Sigma_b. \end{aligned} \tag{6.4}$$

In its last form (6.4) holds also on Σ_{w0}, since both sides vanish in this plane. Substituting in (6.3) yields the expression

$$\begin{aligned} DA &= \oint_{\Sigma_b+\Sigma_{w0}} [(\Phi_{1T} + U\Phi_{1X} + \Phi_{0Z}\Phi_{1Z})\,dY - \Phi_{0Y}\,\Phi_{1Y}\,dZ] \\ &= -\frac{1}{\rho}\mathscr{L}(X,T) - \oint_{\Sigma_b+\Sigma_{w0}} [(\Phi_{0Y}\,\Phi_{1Y} - V\Phi_{0Z})\,dY + \Phi_{0Y}\,\Phi_{1Z}\,dZ]. \end{aligned} \tag{6.5}$$

To show that the last integral vanishes, we first rewrite the boundary condition for Φ_1 on Σ_b in the form

$$\begin{aligned} V\Phi_{0Z}\,dY &= \Phi_{0Z}\,\Phi_{1Z}\,dY + \Phi_{0Z}\,\Phi_{1Y}(F_Y/F_Z)\,dY \\ &= \Phi_{0Z}\,\Phi_{1Z}\,dY - \Phi_{0Z}\,\Phi_{1Y}\,dZ. \end{aligned} \tag{6.6}$$

Since Φ_{0Z} and dZ vanish on Σ_{w0}, this relation can be used on $\Sigma_b + \Sigma_{w0}$ to give

$$\begin{aligned} DA + \frac{1}{\rho}\mathscr{L} &= -\oint_{\Sigma_b+\Sigma_{w0}} [(\Phi_{0Y}\,\Phi_{1Y} - \Phi_{0Z}\,\Phi_{1Z})\,dY + (\Phi_{0Y}\,\Phi_{1Z} + \Phi_{0Z}\,\Phi_{1Y})\,dZ] \\ &= -\iint_{E_x} \left[\frac{\partial}{\partial Z}(\Phi_{0Y}\,\Phi_{1Y} - \Phi_{0Z}\,\Phi_{1Z}) + \frac{\partial}{\partial Y}(\Phi_{0Y}\,\Phi_{1Z} + \Phi_{0Z}\,\Phi_{1Y})\right] dY\,dZ. \end{aligned} \tag{6.7}$$

Here E_x is the domain exterior to the contour $\Sigma_b + \Sigma_{w0}$, and we have used the facts that Φ_0 and Φ_1 are regular outside this contour and the products of their first derivatives vanish more rapidly than $|Y + iZ|^{-1}$ at infinity. Performing the indicated differentiations and invoking Laplace's equation, the integrand of the last integral vanishes, and we obtain the desired result

$$\begin{aligned} \mathscr{L} &= -\rho DA \\ &= -\rho D\oint_{\Sigma_b+\Sigma_w} \Phi_1\,dY. \end{aligned} \tag{6.8}$$

To express this in terms of the far-field behaviour of the velocity potential, we now employ the complex potential (2.26), noting once again that (6.8) contains only first-order terms of $O(h)$. Thus we replace (6.8) by the equation

$$\mathscr{L} = -\rho D\oint_{\Sigma_b+\Sigma_w} \phi\,dy. \tag{6.9}$$

Then, since $f(\zeta) = \phi + i\psi$ satisfies the symmetry condition (2.27), it follows that

$$\oint \phi\, dz = \oint \psi\, dy = 0, \tag{6.10}$$

whereas, from the boundary condition (2.29) and the fact that the stream function is continuous across Σ_w,

$$\begin{aligned}\oint_{\Sigma_b+\Sigma_w} \psi\, dz = \oint_{\Sigma_b} \psi\, dz &= \oint_{\Sigma_b} [-Vy+\psi_0(x,t)]\, dz \\ &= -VS(x),\end{aligned} \tag{6.11}$$

where S is the cross-sectional area of the body. Using these results, it follows that

$$\mathscr{L} = -\rho D \oint_{\Sigma_b+\Sigma_w} f(\zeta)\, d\zeta + \rho D(VS). \tag{6.12}$$

Since f is analytic outside $\Sigma_b+\Sigma_w$, this contour can be deformed to a new one at large ζ, where f possesses a Laurent expansion of the form

$$f = i \sum_{n=1}^{\infty} d_n(x,t)\, \zeta^{-n}, \tag{6.13}$$

and the coefficients $d_n(x,t)$ are all real, by virtue of the symmetry relation. Finally, by residue theory,

$$\mathscr{L}(x,t) = 2\pi\rho D d_1(x,t) + \rho D(VS). \tag{6.14}$$

In order to use (6.14), we return to the solution $f = f_e + f_c$, which was derived in §§3 and 4. The dipole moment associated with f_e is, from (3.12) and (3.5),

$$\begin{aligned} d_{1e} &= \lim_{\zeta\to\infty} -i\zeta f_e(\zeta) = \lim_{\eta\to\infty} -i\eta f_e(\eta) \\ &= V(x,t)\,[c_1(x) - \tfrac{1}{2}\beta^2(x)]. \end{aligned} \tag{6.15}$$

Thus, on leading-edge regions where $f_c = 0$, the differential lift force is

$$\mathscr{L}_e(x,t) = 2\pi\rho D[V(c_1 - \tfrac{1}{2}\beta^2)] + \rho D(VS). \tag{6.16}$$

Alternatively, we may recall (cf. Batchelor 1967, p. 403) that the dipole moment d_{1e} is related to the added-mass coefficient $m(x)$ for lateral acceleration of the body contour Σ_b by the relation

$$2\pi d_{1e} = V(S + m/\rho), \tag{6.17}$$

so that

$$\mathscr{L}_e = -D(mV), \tag{6.18}$$

in agreement with the earlier results of Lighthill (1960). The dipole moment associated with the complementary solution f_c is, from (3.19),

$$d_{1c} = -\frac{1}{\pi}\left(\int_{-\alpha_1}^{-\beta} + \int_{\beta}^{\alpha_2}\right) \frac{\eta_1\, d\eta_1}{(\eta_1^2-\beta^2)^{\frac{1}{2}}}\, \phi_*(\eta_1, x, t). \tag{6.19}$$

The corresponding contribution to the differential lift force is

$$\mathscr{L}_c(x,t) = 2\rho D\left[\left(\int_{-\alpha_1}^{-\beta} + \int_{\beta}^{\alpha_2}\right) \frac{\eta_1\, d\eta_1}{(\eta_1^2-\beta^2)^{\frac{1}{2}}}\, \phi_*(\eta_1, x, t)\right]. \tag{6.20}$$

Equation (6.20) yields the differential lift force due to the presence of trailing vortex sheets. It is valid for quite general bodies with abrupt or slant trailing edges, but if the values of ϕ_* are used from § 4, the results are limited thereby to the abrupt trailing-edge configuration. In any event the value of $\phi_*(\eta_1, x, t)$ can only be computed if the stretched-straight streamlines (U, ϕ_{0y}) are known on the plane $z = 0$, and this restricts our subsequent discussion to simple body forms, such as planar foils without thickness (where the stretched-straight streamlines are parallel to the x axis), and axisymmetric thickness distributions (where slender-body theory yields the value of ϕ_{0y} in a relatively simple form). These specific cases will be discussed further in § 7.

Before leaving the more general discussion of (6.20), it may be compared with the simpler equation of Lighthill (1970, equation 25), which in the present notation may be written as

$$\mathscr{L}_c(x,t) = -D[\tilde{m}(x)\, V(x_*, t_*)]. \tag{6.21}$$

Here $\tilde{m}(x)$ is defined by Lighthill as "the virtual mass associated with the vortex sheet in the presence of a completely stationary cylinder C_x" (i.e. the body contour Σ_b). Lighthill does not evaluate $\tilde{m}(x)$, except in the special case of a flat-plate body without thickness. From the Kutta condition (4.1) and our equation (6.20), it follows that, for a body with a single or symmetric pair of abrupt trailing edges at $x = x_*$, Lighthill's parameter $\tilde{m}(x)$ may be evaluated from the formula

$$\tilde{m}(x) = 2\rho\left(\int_{-\alpha_1}^{-\beta} + \int_{\beta}^{\alpha_1}\right)\frac{(\beta_*^2 - \eta_*^2)^{\frac{1}{2}}}{(\eta_1^2 - \beta^2)^{\frac{1}{2}}}\eta_1\, d\eta_1. \tag{6.22}$$

However, we emphasize that, in general, this integral will depend not only on the longitudinal position x at which the lift force is to be determined, but also on the entire body geometry between this position and the trailing edge at x_*, and, in particular, on the body shape at x_*. (In the special case of a body with axisymmetric thickness and abrupt trailing edges, as will be shown in § 7, the integral (6.22) depends only on the body shape at x and x_*.)

7. Discussion and illustrative examples

In the preceding sections we have outlined a consistent slender-body theory for lateral motions of slender fish-like bodies having both thickness and planar lifting appendages, with particular emphasis on the interaction between the body thickness and lifting effects. This interaction is of importance only for those regions of the body where the trailing vortices shed from the trailing edges exist outboard of the body sections with non-zero thickness, but in these regions it is necessary to account for the deformation of the trailing vortices by the body-thickness-induced streamlines. Indeed, one consequence of this is that the vortex shed at the intersection of the trailing edge and the body will follow the body surface downstream, whereas in the earlier model of Wu & Newman (1972), in which the trailing vortices remained parallel to the free stream, gaps resulted between the inboard vortex and the body surface. Physical considerations suggest that the gap model may be more appropriate for a relatively flat body with small amounts of thickness, since in the limiting case of a flat planar foil the trailing vortices will indeed be parallel to the x axis; but, on the other hand, for a body with substantial thickness the present approach appears to be more consistent both from the mathematical standpoint (i.e. the cross-flow terms in Bernoulli's equation are of comparable magnitude to the usual linear derivatives in t and x) and from the physical standpoint, where Kelvin's theorem suggests that the body surface is a material surface and hence must coincide with any free vorticity which is originally on this surface. Transition between these two complementary approaches will presumably occur as the size of the body thickness is changed, and the question of how thick the body must be may be regarded as analogous to the question of how sharp the trailing edge must be in order to impose a Kutta condition.

The calculation of the differential lift force in § 6 is the most important practical result of our analysis. It is shown that, upstream of the first trailing edges, the differential lift force is proportional to the rate of change of the product of the virtual mass and body velocity, as was well known from earlier studies. But on subsequent downstream portions of the body, at or beyond trailing edges and including the possibility of new leading edges, an additional lift force results and can be interpreted physically as the interaction of the body section with the downwash induced by the trailing vortices. (For a flat plate with constant angle of attack, it is precisely this downwash effect which results in zero loading on the low-aspect-ratio foil downstream of the maximum span position. Indeed, in a branch of lifting-surface theory where the role of the Kutta condition is rarely explicit, we emphasize that this unloading is a logical consequence of imposing the appropriate Kutta condition at the trailing edge.)

As a relatively simple illustration of our results, we shall evaluate the total lift force on a finned body of revolution, with abrupt trailing edges, for the case of steady motion with constant angle of attack. Thus, with $V(x,t) = \text{constant}$, and using† (4.1) to evaluate ϕ_*, equations (6.18) and (6.20) may be integrated along the body length to give the total lift force

$$L = -UVm(l_T) - 2\rho UV\left[\left(\int_{-\alpha_1}^{-\beta} + \int_{\beta}^{\alpha_2}\right)\frac{(\beta_*^2-\eta_*^2)^{\frac{1}{2}}}{(\eta_1^2-\beta^2)^{\frac{1}{2}}}\eta_1\,d\eta_1\right], \tag{7.1}$$

where in the last integral the parameters $\alpha_{1,2}$ and β are evaluated at the trailing edge $x = l_T$, and the added mass is assumed to vanish at the nose $x = -l_N$. For a slender body of revolution, with radius $R = r(x)$, the streamlines $(U, \phi_{0y}, \phi_{0x})$ are determined by the stream function $\psi = -\frac{1}{2}U(R^2 - r^2)$, and hence (η_*, x_*) and (η, x) are related by the streamline equation $R^2 - r^2 = \text{constant}$.

† It should be noted that the use of (4.1) to evaluate ϕ_* on the tail is valid if, and only if, the tail fin span is less than the width of the vortex sheet shed from the upstream fins. In the converse case, however, we have by definition that $\alpha_i = \beta$, and hence the integral in (7.1) vanishes identically. Thus for a body with 're-entrant' tail fins, of span greater than the width of the upstream trailing vortex sheets, the lift force is identical to the classical result for a body with a single pair of fins at the tail; in other words, the vortex sheets generated by the upstream lifting surfaces are completely 're-absorbed' by the tail fins.

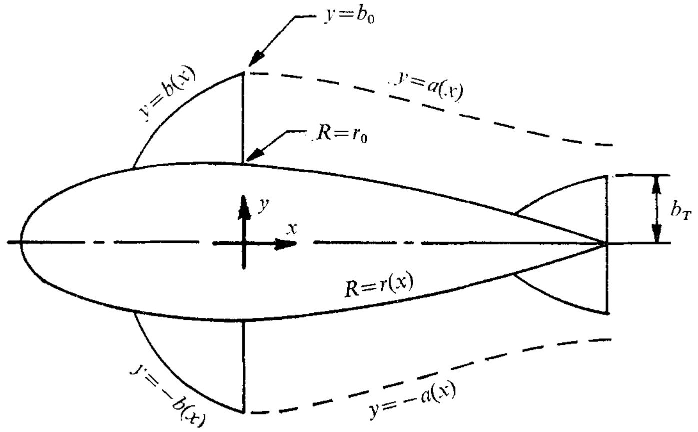

FIGURE 3. The axisymmetric body with symmetric fins.

First, we shall assume the appendages to be symmetrical, as shown in figure 3, with leading edges at $y = \pm b(x)$ and the first trailing edges at $x = 0$, and the tail-fin trailing edges at $x = l_T$. The appropriate (circle) conformal transformation corresponding to (3.5) is

$$\eta = \zeta + r^2/\zeta, \tag{7.2}$$

where

$$\beta = \begin{cases} 2r & \text{on the body,} \\ b + r^2/b & \text{on the fins.} \end{cases}$$

The subsequent formulae are simplified by introducing the notation

$$r(0) \equiv r_0, \quad b(0) \equiv b_0, \quad b(l_T) \equiv b_T,$$

and

$$\alpha_1(l_T) = \alpha_2(l_T) = \alpha_T,$$

where $\alpha_1 = \alpha_2$ by virtue of the fin symmetry.

From (7.2) it follows that

$$\eta_* = y_* + r_0^2/y_* \tag{7.3}$$

and thus

$$\beta_* = b_0 + r_0^2/b_0. \tag{7.4}$$

At the tail, since the body radius vanishes, we have

$$\eta = y \tag{7.5}$$

and

$$\beta = b_T. \tag{7.6}$$

Finally, from the streamline equation,

$$y_*^2 - r_0^2 = y^2, \tag{7.7}$$

where y is evaluated at the tail. From (7.5) and (7.7) the parameter α_T is determined by the equation

$$\alpha_T^2 = b_0^2 - r_0^2. \tag{7.8}$$

Using (7.3) and (7.4) to evaluate $\beta_*^2 - \eta_*^2$, equation (7.1) then takes the form

$$L = -\pi\rho U V b_T^2 - 2\rho U V \left(\int_{-\alpha_T}^{-b_T} + \int_{b_T}^{\alpha_T}\right) \left(\frac{(\alpha_T^2 - \eta_1^2)(\eta_1^2 + \alpha_T^2 r_0^2 b_0^{-2})}{(\eta_1^2 - b_T^2)(\eta_1^2 + r_0^2)}\right)^{\frac{1}{2}} \eta_1 \, d\eta_1, \tag{7.9}$$

where the relation $m(l_T) = \pi\rho b_T^2$ has been used for the added mass of the tail fin. Changing the dummy variable η_1 to v by means of the equation

$$\eta_1^2 = \tfrac{1}{2}(\alpha_T^2 - b_T^2)\,v + \tfrac{1}{2}(\alpha_T^2 + b_T^2) \tag{7.10}$$

permits (7.9) to be written in the simpler form

$$L = -\pi\rho U V b_T^2 - \rho U V(\alpha_T^2 - b_T^2)\int_{-1}^{1}\left(\frac{1-v}{1+v}\frac{\lambda+v}{\mu+v}\right)^{\frac{1}{2}} dv, \tag{7.11}$$

where $$\lambda = (2\alpha_T^2 r_0^2 b_0^{-2} + \alpha_T^2 + b_T^2)/(\alpha_T^2 - b_T^2)$$

and $$\mu = (2r_0^2 + \alpha_T^2 + b_T^2)/(\alpha_T^2 - b_T^2).$$

Equation (7.11) is the final result expressing the lift force on the symmetrical finned body of revolution. In the special case $r_0 = 0$, $\alpha_T = b_0$, while $\mu = \lambda$, and the integral in (7.11) is equal to π, so that the classical flat-plate, low-aspect-ratio lift force $L = -\pi\rho U V b_0^2$ is recovered, confirming that the lift is proportional to the square of the maximum span. (Recalling the footnote on page 688, it follows that $L = -\pi\rho U V b_T^2$ if $b_T > b_0$.) The other special case for which (7.11) can be evaluated analytically is $b_0 = r_0$, or a body without upstream fins; in this case it follows from the definition of α_i that $\alpha_T = b_T$, and hence $L = -\pi\rho U V b_T^2$, again in accordance with the classical results (cf. Thwaites 1960, p. 452).

For more general configurations, (7.11) can be expressed in terms of complete elliptic integrals of the first, second and third kind, but this reduction is cumbersome and will not be repeated here. Numerical integration is straightforward, however, and the results are shown in figure 4. Here the total lift force is non-dimensionalized in terms of the factor $\pi\rho U V b_0^2$, resulting in a lift coefficient which would be equal to unity for a flat delta wing of span $2b_0$. Figure 4 shows this lift coefficient as a function of the parameter r_0/b_0, or the ratio of body radius to semi-span at the first trailing edge. Curves are plotted for different values of the tail-span ratio b_T/b_0, ranging from $b_T = 0$ for a body without tail fins to a maximum value of $b_T = (b_0^2 - r_0^2)^{\frac{1}{2}}$. Beyond the latter limit, or above the dashed curve in figure 4, the tail fins are 're-entrant' and, as noted in the footnote on page 688, the lift coefficient then depends only on the square of the tail semi-span b_T, and is independent of r_0/b_0, as shown by the horizontal portions of each curve. With the exception of this re-entrant regime, we see that the tail fins have a relatively small effect on the total lift force, but the effects of body thickness are much more important, resulting in a substantial reduction of the total lift force by comparison with the classical results based on the maximum span, or on the added-mass coefficient of the finned-circle at the maximum span position.

As an alternative example, we consider the asymmetric configuration shown in figure 5, where only one upstream fin is present. Equation (7.2) is then replaced by the mapping function

$$\eta = \zeta + c_0 + r^2/\zeta, \tag{7.12}$$

where the equations

$$-\beta = -2r_0 + c_0,$$

$$\beta = b_0 + r_0^2/b_0 + c_0$$

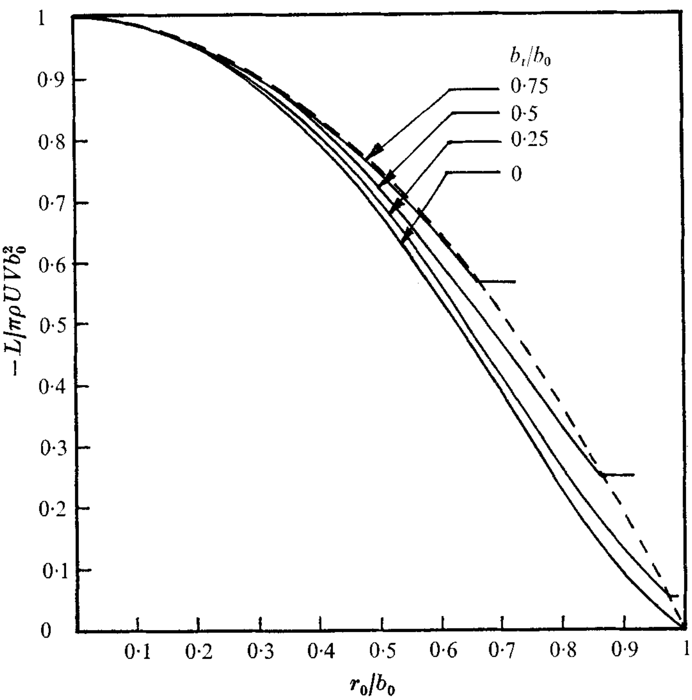

FIGURE 4. Lift coefficient of the axisymmetric body with symmetric fins shown in figure 3. The dashed envelope is the re-entry point where the tail-span equals the width of the vortex sheet, and for larger values of b_T the lift coefficient is independent of b_0 and r_0 as shown.

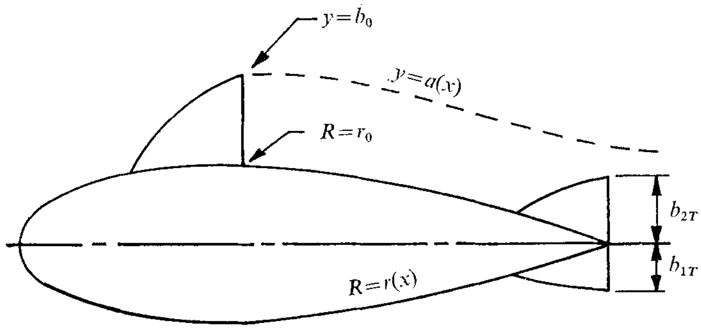

FIGURE 5. The axisymmetric body with asymmetric tail fins and one upstream fin.

determine the parameters β and c_0 at $x = 0$, and $b_0 \equiv b_2(0)$ is the span of the upstream fin. Similarly, at the tail $x = l_T$,

$$\eta = \zeta + c_T, \tag{7.13}$$

where
$$-\beta_T = -b_{1T} + c_T,$$
$$\beta_T = b_{2T} + c_T$$

and $b_{1T} \equiv b_1(l_T)$, $b_{2T} \equiv b_2(l_T)$. Once again the streamlines are determined by the equation $R^2 - r^2(x) = \text{constant}$, and hence in this case it follows that

$$\eta_* = [r_0^2 + (\eta_1 - c_T)^2]^{\frac{1}{2}} + c_0 + r_0^2[r_0^2 + (\eta_1 - c_T)^2]^{-\frac{1}{2}}. \tag{7.14}$$

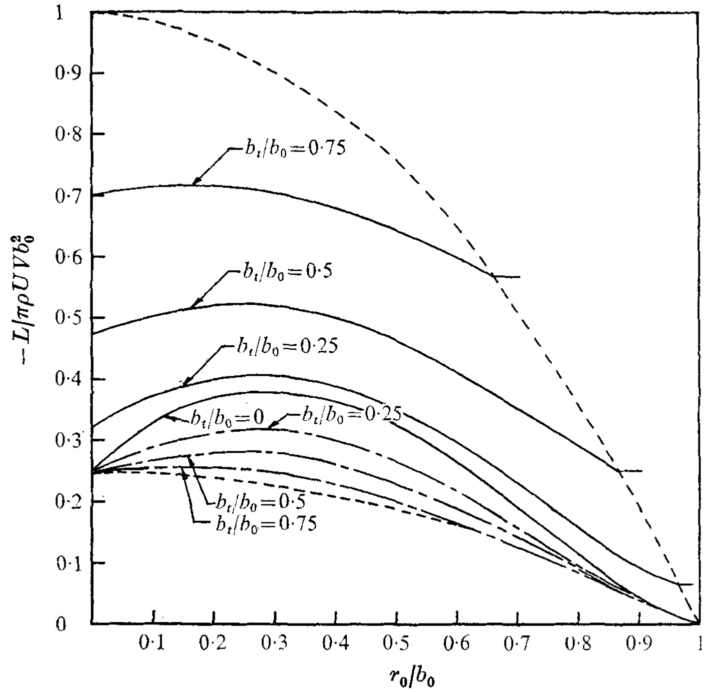

FIGURE 6. Lift coefficient of the axisymmetric body with asymmetric fins shown in figure 5. ——, symmetric tail configuration ($b_{1T} = b_{2T}$); —·—, single upper tail fin ($b_{1T} = 0$). The curve $b_{2T}/b_0 = 0$ is for a body without tail fins. Note that the symmetric tail fin carries a positive lift force, whereas the upper tail fin experiences a negative lift force due to the effects of downwash.

Substituting these results in (7.1) then gives the total lift force in the form

$$L = -\pi\rho U V\beta_T^2 - 2\rho U V\int_{\beta_T}^{\alpha_{2T}} \frac{\eta\, d\eta}{(\eta^2-\beta_T^2)^{\frac{1}{2}}}\{[2r_0+N(\eta)]\,[b_0+r_0^2/b_0-N(\eta)]\}^{\frac{1}{2}}, \quad (7.15)$$

where
$$\beta_T = \tfrac{1}{2}(b_{1T}+b_{2T}),$$
$$\alpha_{2T} = (b_0^2-r_0^2)^{\frac{1}{2}}+\tfrac{1}{2}(b_{1T}-b_{2T})$$
and
$$N(\eta) = \{r_0^2+[\eta-\tfrac{1}{2}(b_{1T}-b_{2T})]^2\}^{\frac{1}{2}}+r_0^2\{r_0^2+[\eta-\tfrac{1}{2}(b_{1T}-b_{2T})]^2\}^{-\frac{1}{2}}.$$

Calculations based on (7.15) are shown in figure 6, for the case of a symmetric tail fin ($b_{1T} = b_{2T}$) by solid lines, and also for a single upper tail fin ($b_{1T} = 0$) by broken lines. The lift coefficient is defined as in figure 4, and plotted as a function of the ratio r_0/b_0 for various tail-fin widths b_{2T}/b_0. For $r_0/b_0 = 0$, or the case of a planar foil, the configuration with $b_{1T} = 0$ agrees with classical low-aspect-ratio theory, the lift being equal to $\frac{1}{4}\pi\rho U V b_0^2$, since in this case b_0 is the maximum span. However, for the symmetrical tail configuration $b_{1T} = b_{2T}$ the classical results are not valid, since there is a new (lower) leading edge at the tail which interacts with the vortex sheet on the opposite side of the foil, so that in this case the lift is not simply related to the total span b_0+b_{1T}, but takes a value slightly less than $\frac{1}{4}\pi\rho U V(b_0+b_{1T})^2$, which would apply if the vortex sheet were filled in by an extension of the foil. Nevertheless, the tail fins do carry a substantial lift force in this case, as compared with the single tail fin ($b_{1T} = 0$) or the symmetrical configuration of figure 4, because here the lower tail fin is free of downwash effects,† as evidenced by the significant increase of lift for increasing values of b_{1T}.

The effects of body thickness, or increasing values of r_0/b_0, are quite striking in figure 6. Here, unlike the symmetric arrangement of figure 4, small or moderate values of body thickness increase the lift force, with a maximum value generally around $r_0/b_0 = 0{\cdot}3$. Thereafter, the lift force decreases, as the fins become small relative to the body thickness, and for $r_0/b_0 = 1$ the limiting case of a bare axisymmetric body without lift is recovered.

The authors gratefully acknowledge support from the National Science Foundation (Grants GK-10846 and GK-10216), and from the Office of Naval Research (Contracts N00014-67-A-0204-0023 and N00014-67-A-0094-0012). The lift coefficients shown in figures 4 and 6 were computed by Mrs Neptune Rodriguez, under support from the Naval Ship Research and Development Center (Contract N 00014-67-A-0204-0058).

REFERENCES

Batchelor, G. K. 1967 *An Introduction to Fluid Mechanics*. Cambridge University Press.

Erdélyi, A. (ed.) 1954 *Tables of Integral Transforms*, vol. II. McGraw-Hill.

Lighthill, M. J. 1960 Note on the swimming of slender fish. *J. Fluid Mech.* **9**, 305–317.

Lighthill, M. J. 1970 Aquatic animal propulsion of high hydromechanical efficiency. *J. Fluid Mech.* **44**, 265–300.

Lighthill, M. J. 1971 Large-amplitude elongated-body theory of fish locomotion. *Proc. Roy. Soc.* B **179**, 125–138.

Milgram, J. H. 1972 Sailing vessels and sails. *Ann. Rev. Fluid Mech.* **4**, 397–430.

Muskhelishvili, N. I. 1953 *Singular Integral Equations*, 2nd edn. (English trans.). Groningen: P. Nordhoff.

Thwaites, B. (ed.) 1960 *Incompressible Aerodynamics*. Oxford University Press.

Timman, R. & Newman, J. N. 1962 The coupled damping coefficients of a symmetric ship. *J. Ship. Res.* **5**, 4, 1–7.

Wu, T. Y. 1971 Hydromechanics of swimming propulsion. Part 3. Swimming and optimum movements of slender fish with side fins. *J. Fluid Mech.* **46**, 545–568.

Wu, T. Y. & Newman, J. N. 1972 Unsteady flow around a slender fish-like body. *Proc. International Symposium on Directional Stability and Control of Bodies Moving in Water*. London: Institution of Mechanical Engineers.

† Physical considerations suggest that, in fact, the lower tail fin will experience an upwash from the upstream vortex sheet.